Exact coherent state transition elements for the squeezed harmonic oscillator

2005 J. Phys. A: Math. Gen. 38 2511

(http://iopscience.iop.org/0305-4470/38/11/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2511–2524 doi:10.1088/0305-4470/38/11/012

Exact coherent state transition elements for the squeezed harmonic oscillator

Richard Sohn[1] **and Mark S Swanson**[2]

[1] 706 Adams St., Holliston, MA 01746, USA
[2] Department of Physics, University of Connecticut, Stamford, CT 06901, USA

Received 5 December 2004, in final form 6 December 2004
Published 2 March 2005
Online at stacks.iop.org/JPhysA/38/2511

Abstract

The squeezed harmonic oscillator Hamiltonian $H = \omega a^\dagger a + \alpha a^2 + \alpha^* a^{\dagger 2}$ is analysed, where $a^\dagger$ and a are harmonic oscillator creation and annihilation operators, ω is real, and α is a time independent constant. For the case that $\omega^2 - 4|\alpha|^2 > 0$ it is known that the Hamiltonian possesses real and positive eigenvalues. The exact eigenstates are constructed by using a Bogoliubov transformation to express them in terms of the algebra and states of the harmonic oscillator. Known properties of these states are reviewed and new results are derived. Modified coherent states are constructed from the exact eigenstates and shown to yield a unit projection operator for the original harmonic oscillator Hilbert space via generalized Gaussian integrals. Transition elements between harmonic oscillator coherent states are then evaluated exactly by using the properties of the modified coherent state projection operator.

PACS numbers: 03.65.−w, 32.90.+a, 42.50.Dv

1. Introduction

The purpose of this paper is to analyse exactly the coherent state [1] transition element for the case that the harmonic oscillator Hamiltonian is generalized to include time independent two photon processes [2], often referred to as the squeezed harmonic oscillator Hamiltonian [3]. Recently Emary and Bishop [4] have used a Bogoliubov transformation [5] to construct the exact energy eigenstates of the closely related one and two-photon Rabi Hamiltonians [6]. This method will be briefly reviewed and then used to find the exact eigenstates for the squeezed harmonic oscillator Hamiltonian in terms of a superposition of the original harmonic oscillator states. Such an expression allows an evaluation of the inner product of the exact eigenstates with the fixed number basis states of the harmonic oscillator. The exact analysis of the harmonic oscillator coherent state transition elements governed by this Hamiltonian is performed by defining modified coherent states written in terms of the exact energy eigenstates and using them to define a unit projection operator over the original harmonic oscillator Hilbert space. Because such a unit projection is a new tool for use in this analysis, its properties are carefully investigated. It is shown that its completeness involves a more complicated pair of Gaussian integrations than those associated with the usual harmonic oscillator coherent states, and that certain criteria must be met to ensure that these Gaussian integrations remain well defined. It is then shown that the transition elements between harmonic oscillator coherent states considered in this paper meet these criteria. The coherent state transition element can then be analysed exactly in the presence of two photon processes by projecting the harmonic oscillator coherent states onto the modified coherent states and performing the resulting generalized Gaussian integrations. The final result, given by equation (99), displays significantly more complicated time dependence due to the two photon processes.

The squeezed harmonic oscillator Hamiltonian can be expressed in terms of bosonic harmonic oscillator creation and annihilation operators as

$$H = \omega a^{\dagger} a + \alpha a^2 + \alpha^* a^{\dagger 2} + \tfrac{1}{2}\omega, \tag{1}$$

where $a^{\dagger}$ and a obey the usual harmonic oscillator commutation relationship $[a, a^{\dagger}] = 1$ and ω is the associated frequency. It is usually assumed that the time-independent constant α is real, but for the purposes of this paper it will be allowed to be complex; the ramifications of this are discussed at the end of section 3. Natural units will be used throughout the paper, so that $\hbar$ and c are set to unity. The operator a annihilates the harmonic oscillator ground state $|0\rangle$, so that $a|0\rangle = 0 \Longrightarrow \langle 0|a^{\dagger} = 0$. The set of states given by

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \tag{2}$$

are eigenstates of the usual harmonic oscillator Hamiltonian $H_0 = \omega a^{\dagger} a + \frac{1}{2}\omega$ and are orthonormal and complete. In this context, it is possible to view the harmonic oscillator states as representing a single frequency mode ω of the photon field, where n determines the number of photons in the state. The terms in the Hamiltonian (1) proportional to α then model direct two photon emission and absorption processes, a fact easily seen by considering the first-order perturbative matrix elements for H. The two-photon Rabi Hamiltonian [6] couples the photon modes of (1) with a two level atom via the Pauli spin matrices. The eigenstates of such a model have been discussed in previous works [4, 7], and so these models will not be considered here. Instead, transition elements for harmonic oscillator coherent states will be calculated exactly for the case that time evolution is governed by the squeezed harmonic oscillator Hamiltonian (1).

The outline of the paper is as follows. In section 2 the Hamiltonian (1) is analysed using operator techniques to derive the structure of its Hilbert space and the inner product of the exact eigenstates with the harmonic oscillator basis states. In section 3 relevant aspects of coherent states are briefly reviewed and the coherent state transition element governed by the Hamiltonian (1) is defined and evaluated exactly using a modified coherent state projection operator defined in terms of the exact eigenstates.

2. Exact energy eigenstates and inner products

In this section the exact energy eigenstates and their inner products will be derived from the harmonic oscillator states and algebra via a Bogoliubov transformation [5]. The general nature of these eigenstates has been discussed elsewhere [4]. However, the results in this section regarding the nature of the inner product between the exact eigenstates and the original harmonic oscillator states are new. It will be seen that these results are essential to verifying completeness of the exact eigenstates and understanding the modified coherent states defined in section 3.

The first step in evaluating (1) is to find its exact eigenstates. This process begins by introducing new operators, c and $c^\dagger$, by means of a Bogoliubov transformation [4, 5]

$$c^\dagger = g_1 a^\dagger - g_2 a \tag{3}$$

$$c = g_1^* a - g_2^* a^\dagger, \tag{4}$$

where in the general case g_1 and g_2 are complex numbers. Demanding that $[c, c^\dagger] = 1$ and using the algebra of the a give

$$[c, c^\dagger] = |g_1|^2 - |g_2|^2 = 1. \tag{5}$$

Result (5) guarantees that definitions (3) and (4) are invertible, yielding

$$a = g_1 c + g_2^* c^\dagger \tag{6}$$

$$a^\dagger = g_1^* c^\dagger + g_2 c. \tag{7}$$

Substituting (6) and (7) into (1) yields

$$H = \Omega c^\dagger c + \beta^* {c^\dagger}^2 + \beta c^2 + \epsilon + \tfrac{1}{2}\omega, \tag{8}$$

where the new coefficients are given in terms of the parameters of (1) and the g_j:

$$\Omega = (|g_1|^2 + |g_2|^2)\omega + 2g_1 g_2^* \alpha + 2g_2 g_1^* \alpha^* \tag{9}$$

$$\beta = g_1 g_2 \omega + {g_2}^2 \alpha^* + {g_1}^2 \alpha \tag{10}$$

$$\epsilon = |g_2|^2 \omega + g_1 g_2^* \alpha + g_2 g_1^* \alpha^*. \tag{11}$$

The process of ordering $c^\dagger$ to the left in all expressions generates the form for ϵ.

The g_1 and g_2 are now chosen so that the complex constraint

$$\beta = \beta^* = 0 \tag{12}$$

is satisfied. If $\alpha = 0$ in (1), then (3) and (4) can be chosen to reduce to $c^\dagger = a^\dagger$ and $c = a$, and this yields the boundary condition

$$\lim_{\alpha \to 0} g_2 = 0. \tag{13}$$

Condition (5) is then consistent with the limit

$$\lim_{\alpha \to 0} g_1 = 1 \tag{14}$$

which will be true up to an arbitrary phase. Condition (5) and the complex constraint (12) remove three degrees of freedom from the two complex numbers g_1 and g_2, which naïvely possess four degrees of freedom. However, g_1 and g_2 actually possess only three physical parameters due to the freedom to adjust the overall phases of the operators in the definitions (6) and (7). This follows from the fact that the process of normal ordering a transition element will receive non-zero contributions only from terms involving equal numbers of a and $a^\dagger$ operators. As a result, transition elements are invariant under the simultaneous phase transformations $a \to \mathrm{e}^{\mathrm{i}\theta} a$ and $a^\dagger \to \mathrm{e}^{-\mathrm{i}\theta} a^\dagger$. These transformations leave (3) and (4) invariant if the g_j undergo the simultaneous transformations $g_1 \to \mathrm{e}^{\mathrm{i}\theta} g_1$ and $g_2 \to \mathrm{e}^{-\mathrm{i}\theta} g_2$. These transformations allow one additional degree of freedom to be removed from the g_j by an appropriate choice for θ. In effect, this means there is a family of solutions to the Bogoliubov transformation that are equivalent up to the phase angle θ. Fixing the phase of the solution to be unity is similar to gauge fixing in gauge field theory [8] and bears no physical significance.

Because all three physically relevant parameters in the g_j are now fixed by condition (5) and constraint (12), solutions to (5) and (12) consistent with the boundary conditions can be found. Constraint (12) yields the quadratic equation

$$\frac{\beta}{{g_1}^2} = \omega\left(\frac{g_2}{g_1}\right) + \alpha + \alpha^*\left(\frac{g_2}{g_1}\right)^2 = 0. \tag{15}$$

The root consistent with the boundary conditions (13) and (14) is given by

$$\frac{g_2}{g_1} = \frac{-\omega + \sqrt{\omega^2 - 4|\alpha|^2}}{2\alpha^*}. \tag{16}$$

Substituting (16) into condition (5) yields

$$|g_1|^2 = \frac{2|\alpha|^2}{(4|\alpha|^2 - \omega^2 + \omega\sqrt{\omega^2 - 4|\alpha|^2})}, \tag{17}$$

which is consistent with boundary condition (14). Combining result (17) with (16) gives

$$|g_2|^2 = \frac{\omega - \sqrt{\omega^2 - 4|\alpha|^2}}{2\sqrt{\omega^2 - 4|\alpha|^2}}. \tag{18}$$

Combining results (17) and (18) yields

$$|g_1|^2 + |g_2|^2 = \frac{\omega}{\sqrt{\omega^2 - 4|\alpha|^2}}. \tag{19}$$

Combining the complex conjugate of (16) with (17) gives

$$2g_1g_2^*\alpha = \left(-\omega + \sqrt{\omega^2 - 4|\alpha|^2}\right)|g_1|^2 = -\frac{2|\alpha|^2}{\sqrt{\omega^2 - 4|\alpha|^2}}. \tag{20}$$

Finally, using (19) and (20) in (9) and (11) yields

$$\Omega = \sqrt{\omega^2 - 4|\alpha|^2} \tag{21}$$

$$\epsilon = \tfrac{1}{2}(\Omega - \omega) \tag{22}$$

These results show that the method of the Bogoliubov transformation breaks down if $\omega^2 - 4|\alpha|^2 \leqslant 0$. However, for the case that $\omega^2 - 4|\alpha|^2 > 0$, referred to hereafter as the Bogoliubov criterion, the Hamiltonian (1) can be transformed into

$$H = \Omega c^\dagger c + \tfrac{1}{2}\Omega. \tag{23}$$

Coupling result (23) with the algebra (5) allows the eigenstates of the Hamiltonian (1) to be defined in a manner identical to that of the simple harmonic oscillator. Defining the state $|\tilde{0}\rangle$ through the relation

$$c|\tilde{0}\rangle = 0 \tag{24}$$

allows the eigenstates of (1) to be defined as

$$|\tilde{n}\rangle = \frac{1}{\sqrt{n!}}{c^\dagger}^n|\tilde{0}\rangle. \tag{25}$$

Using algebra (5) and property (24) immediately gives

$$H|\tilde{n}\rangle = \left(n + \tfrac{1}{2}\right)\Omega|\tilde{n}\rangle. \tag{26}$$

It follows that the eigenvalues of H are real and positive in a manner structurally identical to the simple harmonic oscillator as long as the Bogoliubov criterion is met.

In order to determine the inner product between the eigenstates defined by (25) and the original harmonic oscillator states (2) it is necessary to construct the state $|\tilde{0}\rangle$ from the states of the harmonic oscillator. For the purposes of this paper the Baker–Campbell–Hausdorff theorem [9] states that if the commutator of two operators $[A, B]$ commutes with both A and B, then

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}\, e^{A}\, e^{B} \tag{27}$$

and if $[A, B]$ commutes with B then

$$A\, e^{\lambda B} = e^{\lambda B}(A + \lambda[A, B]) \tag{28}$$

$$A\, e^{\frac{1}{2}\lambda B^2} = e^{\frac{1}{2}\lambda B^2}(A + \lambda[A, B]B), \tag{29}$$

where λ is an arbitrary c-number parameter. Using (29) shows that

$$c \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right) = (g_1^* a - g_2^* a^\dagger) \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right) = \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right) g_1^* a. \tag{30}$$

It then follows that the state that satisfies (24) is given by

$$|\tilde{0}\rangle = N \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right)|0\rangle, \tag{31}$$

where N will be used to define a unit projection operator by normalizing the state (31). Because state (31) has inherited the algebra of the simple harmonic oscillator by its definition, its inner product with itself and the states (2) can now be calculated.

State (31) will first be normalized by fixing N. This is facilitated by expressing (31) in terms of the harmonic oscillator states of (2) by expanding the exponential to obtain

$$|\tilde{0}\rangle = N \sum_{n=0}^{\infty} \frac{1}{2^n n!}\left(\frac{g_2^*}{g_1^*}a^{\dagger 2}\right)^n |0\rangle = N \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!}\left(\frac{g_2^*}{g_1^*}\right)^n |2n\rangle. \tag{32}$$

Using orthonormality of states (2) it follows that

$$\langle\tilde{0}|\tilde{0}\rangle = |N|^2 \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2}\left(\frac{|g_2|^2}{|g_1|^2}\right)^n. \tag{33}$$

Because $|g_2|^2/|g_1|^2 < 1$ the series is absolutely convergent. Using the binomial expansion of $1/\sqrt{1-x}$ shows that

$$\langle\tilde{0}|\tilde{0}\rangle = |N|^2 \sqrt{\frac{|g_1|^2}{|g_1|^2 - |g_2|^2}} = |N|^2 |g_1|, \tag{34}$$

where (5) has been used. As a result, state (31) is normalized by choosing

$$N = \frac{1}{\sqrt{g_1}}. \tag{35}$$

In order to show that the exact eigenstates (25) span the original Hilbert space of harmonic oscillator states (2) it will be necessary to evaluate their inner product with the harmonic oscillator ground state $|0\rangle$. This is accomplished by first noting that the Baker–Campbell–Hausdorff theorem yields the identity

$$c^\dagger \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right) = (g_1 a^\dagger - g_2 a) \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right) = \exp\left(\frac{1}{2}\frac{g_2^*}{g_1^*}a^{\dagger 2}\right)\left(-g_2 a + \frac{a^\dagger}{g_1^*}\right), \tag{36}$$

where (5) has again been used. The inner product then reduces to evaluating

$$\langle 0|\tilde{n}\rangle = \frac{N}{\sqrt{n!}}\langle 0|\left(-g_2 a + \frac{a^\dagger}{g_1^*}\right)^n|0\rangle. \tag{37}$$

The matrix element can be evaluated to give

$$T_n \equiv \langle 0|\left(-g_2 a + \frac{a^\dagger}{g_1^*}\right)^n|0\rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n!}{2^{n/2}(n/2)!}\left(-\frac{g_2}{g_1^*}\right)^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{38}$$

The proof is inductive, and begins by noting that (38) is correct for $n = \{0, 1, 2, 3, 4\}$ by direct calculation. Assuming that T_n and T_{n-2} are given by (38), the next term T_{n+2} can be evaluated inductively with the result that for n an even integer

$$\begin{aligned} T_{n+2} &= -\frac{g_2}{g_1^*}\langle 0|a\left(-g_2 a + \frac{a^\dagger}{g_1^*}\right)^n a^\dagger|0\rangle \\ &= \frac{(n+2)!}{2^{(n+2)/2}\,((n+2)/2)!}\left(-\frac{g_2}{g_1^*}\right)^{(n+2)/2}, \end{aligned} \tag{39}$$

which completes the proof of (38). Using (38) in (37) yields the final form for the inner product

$$\langle 0|\tilde{n}\rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{N\sqrt{n!}}{2^{n/2}(n/2)!}\left(-\frac{g_2}{g_1^*}\right)^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases} \tag{40}$$

with the complex conjugate of (40) holding for $\langle\tilde{n}|0\rangle$.

The proof of completeness for the exact eigenstates reduces to showing that the projection operator $\hat{P} = \sum_{n=0}^{\infty}|\tilde{n}\rangle\langle\tilde{n}|$ is a unit operator in the Hilbert space of the harmonic oscillator states. The orthonormality of states (25) ensures that the operator $\hat{P}$ is idempotent, $\hat{P}^2 = \hat{P}$, so that $\hat{P}(\hat{P} - \hat{1}) = \hat{0}$. The proof that $\hat{P} = \hat{1}$ over the Hilbert space of harmonic oscillator states is inductive, and begins by noting that (40) gives

$$\langle 0|\hat{P}|0\rangle = |N|^2 \sum_{n=0,2,\ldots}^{\infty} \frac{n!}{2^n\left(\frac{n}{2}!\right)^2}\left(\frac{|g_2|^2}{|g_1|^2}\right)^{\frac{n}{2}} = |N|^2\sqrt{\frac{|g_1|^2}{|g_1|^2 - |g_2|^2}} = 1 \tag{41}$$

where the method employed in summing (33) has been used as well as results (5) and (35). As a result, $\langle 0|\hat{P}|0\rangle = 1$. The projection operator must also preserve orthonormality of the original harmonic oscillator states. Using the tools developed so far it is straightforward to show that

$$\langle 1|\hat{P}|0\rangle = \sum_{n=0}^{\infty}\langle 1|\tilde{n}\rangle\langle\tilde{n}|0\rangle = 0. \tag{42}$$

Result (42) follows from the fact that $\langle 1|\tilde{n}\rangle = 0$ for n even, while $\langle\tilde{n}|0\rangle = 0$ for n odd. The next step is to demonstrate that

$$\langle 2|\hat{P}|0\rangle = \sum_{n=0}^{\infty}\langle 2|\tilde{n}\rangle\langle\tilde{n}|0\rangle = 0. \tag{43}$$

From the definitions of the exact eigenstates it follows that

$$\langle 2|\tilde{n}\rangle = \frac{Nn(n-1)}{g_1^{*2}\sqrt{2(n!)}}\langle 0|\left(-g_2 a + \frac{1}{g_1^*}a^\dagger\right)^{(n-2)}|0\rangle + \frac{N}{\sqrt{2(n!)}}\frac{g_2^*}{g_1^*}\langle 0|\left(-g_2 a + \frac{1}{g_1^*}a^\dagger\right)^n|0\rangle, \tag{44}$$

where the first term on the right-hand side of (44) is present only if $n \geqslant 2$. Result (38) can now be combined with the complex conjugate of (40) to give

$$\sum_{n=0}^{\infty}\langle 2|\tilde{n}\rangle\langle\tilde{n}|0\rangle = \frac{|N|^2 g_2^*}{\sqrt{2}g_1^*} + \frac{|N|^2 g_2^*}{\sqrt{2}g_1^*}\sum_{n=2,4,\ldots}^{\infty}\frac{n!}{2^n((n/2)!)^2}\left(1-\frac{n}{|g_2|^2}\right)\left(\frac{|g_2|^2}{|g_1|^2}\right)^{n/2}. \tag{45}$$

Using the identities

$$\frac{g_2^*}{\sqrt{2}g_1^*}\sum_{n=2,4,\ldots}^{\infty}\frac{n!}{2^n((n/2)!)^2}\left(\frac{|g_2|^2}{|g_1|^2}\right)^{n/2} = \frac{g_2^*}{\sqrt{2}g_1^*}(|g_1|-1) \tag{46}$$

$$\frac{1}{\sqrt{2}g_1^* g_2}\sum_{n=2,4,\ldots}^{\infty}\frac{nn!}{2^n((n/2)!)^2}\left(\frac{|g_2|^2}{|g_1|^2}\right)^{n/2} = \frac{g_2^*}{\sqrt{2}g_1^*}|g_1| \tag{47}$$

shows that (45) vanishes. The preservation of orthonormality can now be extended to all the harmonic oscillator states by using induction.

3. Coherent state transition elements

In order to connect these results to optical processes, the transition element between harmonic oscillator coherent states [1] will be calculated for the case that time evolution is governed by the Hamiltonian (1). Harmonic oscillator coherent states are well understood [10] and the brief review presented at the beginning of this section is simply to establish notation and define the transition element to be analysed. In their simplest canonical form the harmonic oscillator coherent state $|\nu\rangle$ is defined as

$$|\nu\rangle = \mathrm{e}^{\nu a^\dagger - \nu^* a}|0\rangle, \tag{48}$$

where ν is an arbitrary dimensionless complex number. Result (27) shows that

$$\mathrm{e}^{\nu a^\dagger - \mu^* a} = \mathrm{e}^{-\frac{1}{2}\mu^*\nu}\,\mathrm{e}^{\nu a^\dagger}\,\mathrm{e}^{-\mu^* a}. \tag{49}$$

Using (49) the coherent state can be written as

$$|\nu\rangle = \mathrm{e}^{\nu a^\dagger - \nu^* a}|0\rangle = \mathrm{e}^{\nu a^\dagger}|0\rangle\,\mathrm{e}^{-\frac{1}{2}\nu^*\nu} = \sum_{n=0}^{\infty}\frac{\nu^n}{\sqrt{n!}}|n\rangle\,\mathrm{e}^{-\frac{1}{2}\nu^*\nu}, \tag{50}$$

where the property $\exp(-\nu^* a)|0\rangle = |0\rangle$ has been used. It follows that

$$\langle\mu|\nu\rangle = \mathrm{e}^{-\frac{1}{2}|\mu|^2 - \frac{1}{2}|\nu|^2 + \mu^*\nu}. \tag{51}$$

One of the important aspects of coherent states is that they give well-defined expectation values for both the momentum $\hat{p}$ and position $\hat{x}$ operators, $\langle\mu|\hat{x}|\mu\rangle = \frac{1}{2}(\mu^* + \mu)/\sqrt{2m\omega}$ and $\langle\mu|\hat{p}|\mu\rangle = \frac{\mathrm{i}}{2}(\mu^* - \mu)\sqrt{m\omega/2}$, where m and ω are the mass and frequency of the harmonic oscillator (in the case of photon it is necessary [11] to replace $m\omega$ with ω). As a result, they represent wave-packets with well-defined expectation values for both position and momentum. Using (51) shows that the coherent states define a unit projection operator since

$$\int [\mathrm{d}\lambda]\,\langle\mu|\lambda\rangle\langle\lambda|\nu\rangle \equiv \int \frac{\mathrm{d}\lambda_R\,\mathrm{d}\lambda_I}{\pi}\langle\mu|\lambda\rangle\langle\lambda|\nu\rangle = \langle\mu|\nu\rangle, \tag{52}$$

where the integration is over both the real λ_R and imaginary λ_I parts of λ and the range of both integrations is $(-\infty, \infty)$. Demonstrating (52) consists of performing two Gaussian integrations using variants of the standard formula

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}\rho^2(1-\mathrm{i}\zeta)x^2 \pm \eta x} = \sqrt{\frac{2\pi}{\rho^2(1-\mathrm{i}\zeta)}} \exp\left\{\frac{\eta^2}{2\rho^2(1-\mathrm{i}\zeta)}\right\} \tag{53}$$

where ρ and ζ are real while η may be complex.

For the case that the initial and final states are chosen to be coherent states the transition element to be analysed in the remainder of this paper is given by

$$W_{\mathrm{fi}} = \langle \mu | \, \mathrm{e}^{-\mathrm{i}H(a^\dagger, a)T} |\nu\rangle, \tag{54}$$

where H is given by (1) and the coherent states are defined by (48) in terms of the harmonic oscillator basis. The exponentiated creation operator appearing in (50) and implicitly in (54) can be treated as a pseudounitary transformation with the property of translating the annihilation operator, since the Baker–Campbell–Hausdorff theorem result (28) gives

$$\mathrm{e}^{-\nu a^\dagger} a \, \mathrm{e}^{\nu a^\dagger} = a + \nu \tag{55}$$

with a similar expression for the action of the exponentiated annihilation operator on the creation operator

$$\mathrm{e}^{\mu^* a} a^\dagger \, \mathrm{e}^{-\mu^* a} = a^\dagger + \mu^*. \tag{56}$$

As a result, using (55) (56), and (49) allows (54) to be rewritten as

$$\begin{aligned} W_{\mathrm{fi}} &= \mathrm{e}^{-\frac{1}{2}(|\mu|^2+|\nu|^2)} \langle 0| \, \mathrm{e}^{\mu^* a} \, \mathrm{e}^{-iH(a^\dagger, a)T} \, e^{\nu a^\dagger} |0\rangle \\ &= \mathrm{e}^{-\frac{1}{2}(|\mu|^2+|\nu|^2-2\mu^*\nu)} \langle 0| \, \mathrm{e}^{-iH(a^\dagger+\mu^*, a+\nu)T} |0\rangle, \end{aligned} \tag{57}$$

where the properties $\exp(-\nu^* a)|0\rangle = |0\rangle$ and $\langle 0| \exp(-\mu a^\dagger) = \langle 0|$ have again been used. The creation and annihilation operators appearing in the Hamiltonian (1) have been translated by the complex numbers characterizing the coherent states.

Evaluating the transition element (57) can be viewed as simply an exercise in normal ordering. However, the evaluation is greatly simplified by projecting the harmonic oscillator ground state onto the states defined by (25). The projection operator $\hat{P}$ is not useful for the coherent state transition element since it leads to intractable infinite sums. Instead, it is more useful to modify the coherent states (48) to be consistent with the algebra of $c^\dagger$ and c and then to define a unit projection operator in terms of the modified coherent states. It will be seen that using such a coherent state projection operator reduces the transition element to a set of Gaussian integrals, consistent with the quadratic nature of (1). The modified coherent states are defined in analogy with (48) as

$$|\tilde{\nu}\rangle = \mathrm{e}^{\nu c^\dagger - \nu^* c} |\tilde{0}\rangle = \sum_{n=0}^{\infty} \frac{\nu^n}{\sqrt{n!}} |\tilde{n}\rangle \, \mathrm{e}^{-\frac{1}{2}\nu^*\nu} \tag{58}$$

$$\langle \tilde{\mu}| = \langle \tilde{0}| \mathrm{e}^{\mu^* c - \mu c^\dagger} = \sum_{n=0}^{\infty} \frac{(\mu^*)^n}{\sqrt{n!}} \langle \tilde{n}| \, \mathrm{e}^{-\frac{1}{2}\mu^*\mu}. \tag{59}$$

These modified coherent states can now be used to define a unit projection operator

$$\hat{P} = \int \frac{\mathrm{d}\nu_I \, \mathrm{d}\nu_R}{\pi} |\tilde{\nu}\rangle\langle\tilde{\nu}| \equiv \int [\mathrm{d}\nu] \, |\tilde{\nu}\rangle\langle\tilde{\nu}| = \hat{1}. \tag{60}$$

The demonstration of (60) contains an important subtlety regarding Gaussian integrations. Using (40) and definition (58) gives

$$\langle 0|\tilde{\nu}\rangle = N \sum_{n=0,2,\ldots}^{\infty} \frac{\nu^n}{2^{n/2}(n/2)!} \left(-\frac{g_2}{g_1^*}\right)^{n/2} \mathrm{e}^{-\frac{1}{2}\nu^*\nu} = N \exp\left(-\frac{\nu^2 g_2}{2g_1^*} - \frac{1}{2}\nu^*\nu\right). \tag{61}$$

In an identical manner it follows that

$$\langle\tilde{\nu}|0\rangle = N^* \exp\left(-\frac{\nu^{*2} g_2^*}{2g_1} - \frac{1}{2}\nu^*\nu\right). \tag{62}$$

For the non-interacting case that $g_2 \to 0$ and $g_1 \to 1$ both results (61) and (62) reduce to the form given by (51). However, it is important to determine under what conditions the presence of the additional quadratic term in (62) is consistent with the same measure of integration as (52). Denoting $\nu = a + b\mathrm{i}$, it follows that

$$\int [\mathrm{d}\nu]\, \langle 0|\tilde{\nu}\rangle\langle\tilde{\nu}|0\rangle = |N|^2 \int \frac{\mathrm{d}a\,\mathrm{d}b}{\pi} \exp\{-a^2(1+q_1) - b^2(1-q_1) + 2q_2 ab\mathrm{i}\} \tag{63}$$

where

$$q_1 = \frac{1}{2}\left(\frac{g_2^*}{g_1} + \frac{g_2}{g_1^*}\right) \tag{64}$$

and

$$q_2 = \frac{1}{2}\left(\frac{g_2^*}{g_1} - \frac{g_2}{g_1^*}\right). \tag{65}$$

While q_2 is pure imaginary, the expression q_1 is real. If $|q_1| \geqslant 1$, then one of the Gaussian integrations in (63) will be divergent and (60) will no longer be unity. It is therefore critical to determine the conditions for which this does not occur. Expressing the two complex parameters in the polar form, $g_1 = |g_1|\,\mathrm{e}^{ig}$ and $\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}z}$, and using (16) and (17), it is straightforward to show that

$$q_1 = \left(\frac{\Omega - \omega}{2|\alpha|}\right)\cos(2g + z). \tag{66}$$

Using the form for Ω shows that $|q_1| < 1$ as long as $\omega > 2|\alpha|$. This is the same criterion determined earlier for the validity of the Bogoliubov transformation. As a result, as long as the Bogoliubov criterion is met, the Gaussian integrations in (63) will be well defined. Assuming this criterion is met, the b integration in the coherent state projection operator (63) then yields

$$\int [\mathrm{d}\nu]\, \langle 0|\tilde{\nu}\rangle\langle\tilde{\nu}|0\rangle = \frac{|N|^2}{\sqrt{1-q_1}} \int \frac{\mathrm{d}a}{\sqrt{\pi}} \exp\left\{-a^2\left(\frac{1-q_1^2+q_2^2}{1-q_1}\right)\right\}. \tag{67}$$

Using the definitions (64) and (65) as well as condition (5) immediately yields

$$1 - q_1^2 + q_2^2 = 1 - \frac{|g_2|^2}{|g_1|^2} = \frac{1}{|g_1|^2} > 0. \tag{68}$$

It follows that the remaining Gaussian integration in (67) is also well defined since both the numerator and denominator of $\left(1 - q_1^2 + q_2^2\right)\big/(1 - q_1)$ are positive. The final result of integrating a in (67) is therefore

$$\int [\mathrm{d}\nu]\, \langle 0|\tilde{\nu}\rangle\langle\tilde{\nu}|0\rangle = |N|^2|g_1| = 1 \tag{69}$$

demonstrating that the normalization (35) yields a unit coherent state projection operator. While they are algebraically more complicated, higher-order matrix elements may be demonstrated similarly.

The Hamiltonian appearing in the matrix element (54) will now be expressed in terms of the $c^\dagger$ and c operators. Using forms (3) and (4) it follows that the translated Hamiltonian takes the form

$$H(a^\dagger + \mu^*, a + \nu) = \Omega c^\dagger c + \gamma c^\dagger + \delta c + K + \tfrac{1}{2}\Omega \equiv H_{\mathrm{eff}}(c^\dagger, c), \tag{70}$$

where γ, δ and K are given by

$$\gamma = (\omega g_2^* + 2g_1^*\alpha^*)\mu^* + (\omega g_1^* + 2g_2^*\alpha)\nu = \Omega(g_1^*\nu - g_2^*\mu^*) \tag{71}$$

$$\delta = (\omega g_1 + 2g_2\alpha^*)\mu^* + (\omega g_2 + 2g_1\alpha)\nu = \Omega(g_1\mu^* - g_2\nu) \tag{72}$$

$$K = \Omega(g_1\mu^* - g_2\nu)(g_1^*\nu - g_2^*\mu^*) = \frac{\gamma\delta}{\Omega} \tag{73}$$

and the properties of the g_j have been used. The matrix element (57) can now be evaluated by projecting the state $|0\rangle$ onto the modified coherent states

$$W_{fi} = \int [\mathrm{d}\sigma][\mathrm{d}\tau]\langle 0|\tilde{\sigma}\rangle\langle\tilde{\sigma}|\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{eff}}(c^\dagger,c)T}\,|\tilde{\tau}\rangle\langle\tilde{\tau}|0\rangle\,\mathrm{e}^{-\frac{1}{2}(|\mu|^2+|\nu|^2-2\mu^*\nu)}. \tag{74}$$

Similarly to (50) the modified coherent states are written as

$$\langle\tilde{\sigma}| = \mathrm{e}^{-\frac{1}{2}\sigma^*\sigma}\langle\tilde{0}|\,\mathrm{e}^{\sigma^* c} \tag{75}$$

$$|\tilde{\tau}\rangle = \mathrm{e}^{\tau c^\dagger}|\tilde{0}\rangle\,\mathrm{e}^{-\frac{1}{2}\tau^*\tau} \tag{76}$$

so that the exponentiated operators play a role for the c algebra identical to that of (55) and (56) for the a algebra. Combining this property with (49), (61) and (62), (74) becomes

$$W_{fi} = |N|^2\int [\mathrm{d}\sigma][\mathrm{d}\tau]\,\mathrm{e}^{-(|\sigma|^2+|\tau|^2-\sigma^*\tau+\frac{g_2^*}{2g_1}\tau^{*2}+\frac{g_2}{2g_1^*}\sigma^2)}\langle\tilde{0}|\,\mathrm{e}^{-iH_{\mathrm{eff}}(c^\dagger+\sigma^*,c+\tau)T}\,|\tilde{0}\rangle\,\mathrm{e}^{-\frac{1}{2}(|\mu|^2+|\nu|^2-2\mu^*\nu)}, \tag{77}$$

where the Hamiltonian has undergone another translation similar in form to (57) and is given explicitly by

$$H_{\mathrm{eff}}(c^\dagger+\sigma^*, c+\tau) = \Omega c^\dagger c + (\Omega\tau+\gamma)c^\dagger + (\Omega\sigma^*+\delta)c + \Omega\sigma^*\tau + \gamma\sigma^* + \delta\tau + \tfrac{1}{2}\Omega + K. \tag{78}$$

The contribution of the operator part of (78) to the matrix element in (77) can be evaluated exactly by a straightforward application of the Baker–Campbell–Hausdorff theorem. Denoting $\tilde{\gamma} = \Omega\tau + \gamma$ and $\tilde{\delta} = \Omega\sigma^* + \delta$, it follows that

$$\begin{aligned} Z_{fi} &\equiv \langle\tilde{0}|\,\mathrm{e}^{-i(\Omega c^\dagger c+\tilde{\gamma}c^\dagger+\tilde{\delta}c)T}\,|\tilde{0}\rangle \\ &= \langle\tilde{0}|\,\mathrm{e}^{-\tilde{\gamma}c^\dagger/\Omega}\,\mathrm{e}^{\tilde{\delta}c/\Omega}\,\mathrm{e}^{-i\Omega c^\dagger cT}\,\mathrm{e}^{-\tilde{\delta}c/\Omega}\,\mathrm{e}^{\tilde{\gamma}c^\dagger/\Omega}|\tilde{0}\rangle\,\mathrm{e}^{\mathrm{i}\tilde{\gamma}\tilde{\delta}T/\Omega} \\ &= \langle\tilde{0}|\,\mathrm{e}^{\tilde{\delta}c/\Omega}\,\mathrm{e}^{-\mathrm{i}\Omega c^\dagger cT}\,\mathrm{e}^{\tilde{\gamma}c^\dagger/\Omega}|\tilde{0}\rangle\,\mathrm{e}^{-\tilde{\gamma}\tilde{\delta}/\Omega^2}\,\mathrm{e}^{\mathrm{i}\tilde{\gamma}\tilde{\delta}T/\Omega}. \end{aligned} \tag{79}$$

An extension of result (28) yields

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}\Omega c^\dagger cT}c^{\dagger n} = (c^\dagger\,\mathrm{e}^{-\mathrm{i}\Omega T})^n\,\mathrm{e}^{-i\Omega c^\dagger cT} \\ &\Longrightarrow \mathrm{e}^{-\mathrm{i}\Omega c^\dagger cT}\exp(\tilde{\gamma}c^\dagger/\Omega) = \exp(\tilde{\gamma}c^\dagger\,\mathrm{e}^{-\mathrm{i}\Omega T}/\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega c^\dagger cT}. \end{aligned} \tag{80}$$

while (24) ensures that

$$\mathrm{e}^{-\mathrm{i}\Omega c^\dagger cT}|\tilde{0}\rangle = |\tilde{0}\rangle. \tag{81}$$

Using these results in (79) and applying (27) twice yields the final result

$$\begin{aligned} Z_{fi} &= \langle\tilde{0}|\exp(\tilde{\delta}c/\Omega)\exp(\tilde{\gamma}c^\dagger\,\mathrm{e}^{-\mathrm{i}\Omega T}/\Omega)|\tilde{0}\rangle\,\mathrm{e}^{-\tilde{\gamma}\tilde{\delta}/\Omega^2}\,\mathrm{e}^{\mathrm{i}\tilde{\gamma}\tilde{\delta}T/\Omega} \\ &= \exp\left\{-\frac{\tilde{\gamma}\tilde{\delta}}{\Omega^2}\left(1-\mathrm{e}^{-\mathrm{i}\Omega T}\right) + \mathrm{i}\frac{\tilde{\gamma}\tilde{\delta}T}{\Omega}\right\}. \end{aligned} \tag{82}$$

Using result (82) and the definitions of $\tilde{\gamma}$, $\tilde{\delta}$ and K in (77) gives

$$W_{fi} = |N|^2 \exp\left[-\frac{\gamma\delta}{\Omega^2}\left(1 - \mathrm{e}^{-\mathrm{i}\Omega T}\right) - \mathrm{i}\frac{1}{2}\Omega T - \frac{1}{2}(|\mu|^2 + |\nu|^2 - 2\mu^*\nu)\right]$$
$$\times \int [\mathrm{d}\sigma][\mathrm{d}\tau] \exp\left[-|\sigma|^2 - |\tau|^2 - \frac{g_2^*}{2g_1}\tau^{*2} - \frac{g_2}{2g_1^*}\sigma^2 + \sigma^*\tau\,\mathrm{e}^{-i\Omega T} - \left(\frac{\gamma\sigma^*}{\Omega} + \frac{\delta\tau}{\Omega}\right)(1 - \mathrm{e}^{-\mathrm{i}\Omega T})\right]. \tag{83}$$

The evaluation of the matrix element is now reduced to performing four Gaussian integrals of form (53). The computation is tedious but straightforward, and so only the major details will be presented. Suppressing all factors in (83) that are independent of σ and τ gives the Gaussian integrals

$$G = \int [\mathrm{d}\tau] \exp\left\{-|\tau|^2 - \frac{g_2^*}{2g_1}\tau^{*2} - \frac{\delta\tau}{\Omega}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})\right\}$$
$$\times \underbrace{\int [\mathrm{d}\sigma] \exp\left\{-|\sigma|^2 - \frac{g_2}{2g_1^*}\sigma^2 + \sigma^*\tau\,\mathrm{e}^{-\mathrm{i}\Omega T} - \frac{\gamma\sigma^*}{\Omega}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})\right\}}_{\equiv Q}. \tag{84}$$

The integral Q solely over σ is next rewritten using $\sigma = a + b\mathrm{i}$, so that it becomes

$$Q = \int \frac{\mathrm{d}a\,\mathrm{d}b}{\pi} \exp\left\{-(1+q)a^2 - (1-q)b^2 - 2\mathrm{i}qab + 2z(a - b\mathrm{i})\right\}, \tag{85}$$

where

$$q = \frac{g_2}{2g_1^*} \tag{86}$$

$$z = \frac{1}{2}\left[\tau\,\mathrm{e}^{-\mathrm{i}\Omega T} - \frac{\gamma}{\Omega}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})\right]. \tag{87}$$

The integrations over a and b are defined only if the absolute value of the real part of q is less than one. However, from (53) it is clear that these integrations are well defined regardless of the value of the imaginary part of q. Using (16) and the polar decomposition preceding (66) shows that the real part of q is given by

$$\mathrm{Re}\, q = \left(\frac{\Omega - \omega}{4|\alpha|}\right)\cos(2g + z). \tag{88}$$

It is straightforward to show that the absolute value of the real part of q is less than $1/2$ as long as the Bogoliubov criterion $\omega > 2|\alpha|$ is satisfied, so that the integrations in (85) are well defined. The integration over a is then a variant of (53) and yields, after some algebraic manipulation,

$$Q = \sqrt{\frac{\pi}{1+q}} \int \frac{\mathrm{d}b}{\pi} \exp\left\{-\frac{b^2}{1+q} - 2ibz\left(\frac{1+2q}{1+q}\right) + \frac{z^2}{1+q}\right\} \tag{89}$$

Performing the integral over b using (53) then yields, after some algebraic manipulation,

$$Q = \exp\{-4qz^2\} = \exp\left\{-q\left[\tau^2\,\mathrm{e}^{-2\mathrm{i}\Omega T} - \frac{2\tau\gamma}{\Omega}\,\mathrm{e}^{-\mathrm{i}\Omega T}(1 - \mathrm{e}^{-\mathrm{i}\Omega T}) + \frac{\gamma^2}{\Omega^2}\left(1 - \mathrm{e}^{-\mathrm{i}\Omega T}\right)^2\right]\right\} \tag{90}$$

Recombining (90) with (84) yields

$$G = \int [\mathrm{d}\tau] \exp\left\{ -|\tau|^2 - q^*\tau^{*2} - \frac{\delta\tau}{\Omega}(1 - \mathrm{e}^{-\mathrm{i}\Omega T}) - q\tau^2\,\mathrm{e}^{-2\mathrm{i}\Omega T} + \frac{2q\tau\gamma}{\Omega}\,\mathrm{e}^{-\mathrm{i}\Omega T}(1 - \mathrm{e}^{-\mathrm{i}\Omega T}) - \frac{q\gamma^2}{\Omega^2}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})^2 \right\} \tag{91}$$

The integral G is also Gaussian, and the decomposition $\tau = a + b\mathrm{i}$ yields

$$G = \int \frac{\mathrm{d}a\,\mathrm{d}b}{\pi} \exp\left\{ -(1+u)a^2 - (1-u)b^2 + 2a\,(\mathrm{i}bv - w) - 2b\mathrm{i}w \right\}. \times \exp\left[-\frac{q\gamma^2}{\Omega^2}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})^2 \right] \tag{92}$$

where

$$u = q^* + q\,\mathrm{e}^{-2\mathrm{i}\Omega T} \tag{93}$$

$$v = q^* - q\,\mathrm{e}^{-2\mathrm{i}\Omega T} \tag{94}$$

$$w = \left(\frac{1}{2}\frac{\delta}{\Omega} - \frac{q\gamma}{\Omega}\,\mathrm{e}^{-\mathrm{i}\Omega T} \right)(1 - \mathrm{e}^{-\mathrm{i}\Omega T}). \tag{95}$$

It is straightforward to use (16) and the polar decomposition preceding (66) to show that the real part of u is given by

$$\mathrm{Re}\,u = \left(\frac{\Omega - \omega}{4|\alpha|} \right)(\cos(2g + z) + \cos(2g + z - 2\Omega T)). \tag{96}$$

The absolute value of the total trigonometric contribution to (96) is bounded by 2, so that $|\mathrm{Re}\,u| \leqslant (\omega - \Omega)/2|\alpha|$. Once again, this is less than unity if the Bogoliubov criterion is met and the Gaussian integrals therefore remain well defined. Performing the integration over a yields, after some algebraic manipulation,

$$G = \exp\left[-\frac{q\gamma^2}{\Omega^2}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})^2 \right] \sqrt{\frac{\pi}{1+u}} \times \int \frac{\mathrm{d}b}{\pi} \exp\left\{ -\frac{1}{(1+u)} \left[\left(1 - 4q^*q\,\mathrm{e}^{-2\mathrm{i}\Omega T}\right) b^2 - 2\mathrm{i}bw\,(1 + u + v) + w^2 \right] \right\}. \tag{97}$$

Result (97) is yet another Gaussian integral and is well defined since $4q^*q$ is a real positive number less than unity, while result (66) shows that $1/(1+u)$ has a real part that is positive. The evaluation of (97) then yields, after algebraic manipulation,

$$G = \sqrt{\frac{1}{1 - 4q^*q\,\mathrm{e}^{-2\mathrm{i}\Omega T}}} \exp\left\{ -\frac{4q^*w^2}{1 - 4q^*q\,\mathrm{e}^{-2\mathrm{i}\Omega T}} - \frac{q\gamma^2}{\Omega^2}(1 - \mathrm{e}^{-\mathrm{i}\Omega T})^2 \right\}. \tag{98}$$

Using (86) and (95) in (98) and returning the result to (83) yield the final exact result for the transition element

$$W_{fi} = \frac{1}{\sqrt{|g_1|^2 - |g_2|^2\,\mathrm{e}^{-2\mathrm{i}\Omega T}}} \exp\left\{ -\frac{\gamma\delta(1 - \mathrm{e}^{-\mathrm{i}\Omega T})(|g_1|^2 - |g_2|^2\,\mathrm{e}^{-\mathrm{i}\Omega T})}{\Omega^2(|g_1|^2 - |g_2|^2\,\mathrm{e}^{-2\mathrm{i}\Omega T})} - \mathrm{i}\frac{1}{2}\Omega T - \frac{(1 - \mathrm{e}^{-\mathrm{i}\Omega T})^2}{2\Omega^2(|g_1|^2 - |g_2|^2\,\mathrm{e}^{-2\mathrm{i}\Omega T})}(g_1 g_2 \gamma^2 + g_1^* g_2^* \delta^2) - \frac{1}{2}(|\mu|^2 + |\nu|^2 - 2\mu^*\nu) \right\}. \tag{99}$$

All terms in (99) can be expressed uniquely in terms of ω, α, μ and ν. While $|g_1|^2$ and $|g_2|^2$ are given by (17) and (18) respectively, it is straightforward to use their forms in conjunction with (20), (71) and (72) to show that

$$\gamma\delta = \Omega(\omega\mu^*\nu + \alpha^*\mu^{*2} + \alpha\nu^2) \tag{100}$$

$$g_1^* g_2^* \delta^2 = -\frac{1}{2}\left(\frac{\alpha}{\omega - \Omega}\right)[(\omega - \Omega)\nu + 2\alpha^*\mu^*]^2 \tag{101}$$

$$g_1 g_2 \gamma^2 = -\frac{1}{2}\left(\frac{\alpha^*}{\omega - \Omega}\right)[(\omega - \Omega)\mu^* + 2\alpha\nu]^2. \tag{102}$$

Using the results derived so far it is easy to show that in the $\alpha \to 0$ limit

$$\lim_{\alpha\to 0} W_{fi} = \exp\left(-\tfrac{1}{2}|\mu|^2 - \tfrac{1}{2}|\nu|^2 + \mu^*\nu\,\mathrm{e}^{-\mathrm{i}\omega T} - \tfrac{1}{2}\mathrm{i}\omega T\right), \tag{103}$$

which is the well known result for the simple harmonic oscillator in the absence of two photon processes. A particularly simple form for (99) emerges when the vacuum persistence probability is examined. Setting $\mu = \nu = 0$ in (99) yields

$$|\langle 0|\,\mathrm{e}^{-\mathrm{i}HT}|0\rangle|^2 = \frac{1}{\sqrt{|g_1|^4 - 2|g_1|^2|g_2|^2\cos(2\Omega T) + |g_2|^4}}. \tag{104}$$

Using (5) and (19) shows that the vacuum persistence probability oscillates between 1 and Ω/ω with the frequency 2Ω.

Another interesting limit to consider is the case where $\Omega \to 0$, which is the point where the Bogoliubov transformation breaks down. Result (99) is a function of both α and α^* and is therefore not analytic in α. Because α can be complex, there is an infinite set of ways to take the limit $\Omega \to 0$ and the non-analyticity of W_{fi} will therefore yield different results. It is usually the case that α is assumed to be real. The limit $\Omega \to 0$ is then taken along the real axis, so that $\alpha = \alpha^* \to \frac{1}{2}\omega$. It is straightforward to show that (99) then yields

$$\lim_{\alpha\to\frac{1}{2}\omega} W_{fi} = \exp\left\{-\frac{1}{2}\left(\mathrm{i}\omega T + \frac{1}{2}\omega^2 T^2\right)\left(\mu^* + \nu\right)^2 - \frac{1}{2}(|\mu|^2 + |\nu|^2 - 2\mu^*\nu)\right\}. \tag{105}$$

For the case that α is pure imaginary, the limit requires $\alpha = -\alpha^* \to \frac{1}{2}\mathrm{i}\omega$, and this yields

$$\lim_{\alpha\to\frac{1}{2}\mathrm{i}\omega} W_{fi} = \exp\left\{-\tfrac{1}{2}\omega T(\mu^* + \mathrm{i}\nu)^2 - \tfrac{1}{4}\mathrm{i}\omega^2 T^2(\nu - \mathrm{i}\mu^*)^2 - \tfrac{1}{2}(|\mu|^2 + |\nu|^2 - 2\mu^*\nu)\right\}. \tag{106}$$

The two results (105) and (106) are different, bearing out the non-analyticity of the point $\Omega = 0$ in particular. For such a case, it is interesting to consider the coherent state persistence function, so that $\mu = \nu$. In the event that (105) is the relevant $\Omega \to 0$ limit, it follows that the coherent state persistence probability becomes

$$\lim_{\Omega\to 0} |\langle\mu|\,\mathrm{e}^{-\mathrm{i}HT}|\mu\rangle|^2 = \exp\left\{-\tfrac{1}{2}\omega^2 T^2(\mu + \mu^*)^2\right\}. \tag{107}$$

The $\Omega \to 0$ limit corresponds to unstable coherent states, with the exception of the state localized at $\langle x\rangle = 0$, as discussed following (51). Despite instability, result (107) exhibits the general property that the persistence probability is time-reversal invariant under $T \to -T$.

References

[1] Schrödinger E 1926 *Naturwissenschaften* **14** 664
Glauber R 1963 *Phys. Rev.* **131** 2766

[2] Allen L and Eberly J 1975 *Optical Resonance and Two-Level Atoms* (New York: Wiley)
Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1989 *Photons and Atoms* (New York: Wiley)
Yariv A 1989 *Quantum Electronics 3 edn* (New York: Wiley)
[3] Bishop R and Vourdas A 1986 *J. Phys. A: Math. Gen.* **19** 2525
[4] Emary C and Bishop R 2002 *J. Math. Phys.* **43** 3916
Emary C and Bishop R 2002 *J. Phys. A: Math. Gen.* **35** 8231
[5] Bogoliubov N 1947 *J. Phys. (USSR)* **11** 23
[6] Rabi I 1937 *Phys. Rev.* **51** 652
[7] Kùs M and Lewenstein M 1986 *J. Phys. A: Math. Gen.* **19** 305
[8] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Prindeton, NJ: Princeton University Press)
[9] Baker H 1904 *Proc. Lond. Math. Soc. Ser.* **3** 24
Campbell J 1898 *Proc. Lond. Math. Soc.* **29** 14
Hausdorff F 1906 *Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Naturwis. Klasse* **58** 19
[10] Klauder J and Sudarshan E 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
Klauder J and Skagerstam B 1985 *Coherent States* (Singapore: World Scientific)
Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[11] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1989 *Photons and Atoms* (New York: Wiley)
Peskin M and Schroeder D 1995 *An Introduction to Quantum Field Theory* (Reading, MA: Addison-Wesley)